

INVERSE SPECTRAL PROBLEMS FOR SCHRÖDINGER OPERATORS

HAMID HEZARI

ABSTRACT. In this article we improve some of the inverse spectral results proved by Guillemin and Uribe in [GU]. They proved that under some symmetry assumptions on the potential $V(x)$, the Taylor expansion of $V(x)$ near a non-degenerate global minimum can be recovered from the knowledge of the low-lying eigenvalues of the associated Schrödinger operator in \mathbb{R}^n . We prove some similar inverse spectral results using fewer symmetry assumptions. We also show that in dimension 1, no symmetry assumption is needed to recover the Taylor coefficients of $V(x)$. We establish our results by finding some explicit formulas for wave invariants at the bottom of the well.

Keywords: Schrödinger operator, Wave invariants, Semi-Classical trace formulae.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this article we study some inverse spectral problems of the eigenvalue problem for the semi-classical Schrödinger operator,

$$(1) \quad \hat{P} = -\frac{\hbar^2}{2}\Delta + V(x) \quad \text{on } L^2(\mathbb{R}^n),$$

associated to the Hamiltonian

$$P(x, \xi) = \frac{1}{2}\xi^2 + V(x).$$

Here the potential $V(x)$ in (1) satisfies

$$(2) \quad \begin{cases} V(x) \in C^\infty(\mathbb{R}^n), \\ V(x) \text{ has a unique non-degenerate global minimum at } x = 0 \text{ and } V(0) = 0, \\ \text{For some } \varepsilon > 0, V^{-1}[0, \varepsilon] \text{ is compact.} \end{cases}$$

Under these conditions for sufficiently small \hbar , say $\hbar \in (0, h_0)$, and sufficiently small δ , a classical fact tells us the spectrum of \hat{P} in the energy interval $[0, \delta]$ is finite. We denote these eigenvalues by

$$\{E_j(\hbar)\}_{j=0}^m.$$

We call these eigenvalues the low-lying eigenvalues of \hat{P} . We notice the Weyl's law reads

$$(3) \quad m = N_\hbar(\delta) = \#\{j; 0 \leq E_j(\hbar) \leq \delta\} = \frac{1}{(2\pi\hbar)^n} \left(\int_{\frac{1}{2}\xi^2 + V(x) \leq \delta} dx d\xi + o(1) \right).$$

Recently in [GU], Guillemin and Uribe raised the question whether we can recover the Taylor coefficients of V at $x = 0$ from the low-lying eigenvalues $E_j(\hbar)$. They also established that if we assume some symmetry conditions on V , namely $V(x) = f(x_1^2, \dots, x_n^2)$, then the 1-parameter family of low-lying eigenvalues, $\{E_j(\hbar) \mid \hbar \in (0, h_0)\}$, determines the Taylor coefficients of V at $x = 0$.

In this article we will attempt to recover as much of V as possible from the family $E_j(\hbar)$, by establishing some new formulas for the wave invariants at the bottom of the potential (Theorem 1.1). Using these new expressions for the wave invariants, in Theorem 1.2 we improve the inverse spectral results of [GU] for a larger class of potentials.

A classical approach in studying this problem is to examine the asymptotic behavior as $\hbar \rightarrow 0$ of the truncated trace

$$(4) \quad \text{Tr}(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}),$$

where $\Theta \in C_0^\infty([0, \infty))$ is supported in $I = [0, \delta]$ and equals one in a neighborhood of 0.

The asymptotic behavior of the truncated trace $\text{Tr}(t, \hbar)$ around the equilibrium point $(x, \xi) = (0, 0)$ has been extensively studied in the literature. It is known that (see for example [BPU]) for t in a sufficiently small interval $(0, t_0)$, $\text{Tr}(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}})$ has an asymptotic expansion of the following form:

$$(5) \quad \text{Tr}(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) \sim \sum_{j=0}^{\infty} a_j(t) \hbar^j, \quad \hbar \rightarrow 0.$$

Throughout this paper when we refer to wave invariants at the bottom of the well, we mean the coefficients $a_j(t)$ in (5).

By applying an orthogonal change of variable, we can assume that V is of the form

$$(6) \quad V(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W(x), \quad \omega_k > 0,$$

$$W(x) = O(|x|^3), \quad |x| \rightarrow 0.$$

In addition to conditions in (2), we also assume that $\{\omega_k\}$ are linearly independent over \mathbb{Q} . We note that we have $W(0) = \nabla W(0) = \text{Hess}W(0) = 0$.

Our first result finds explicit formulas for the wave invariants.

THEOREM 1.1. *For $0 < t < \min_{1 \leq k \leq n} \{\frac{\pi}{2\omega_k}\}$,*

1.

$$a_0(t) = \text{Tr}(e^{\frac{-it}{\hbar}\hat{H}_0}) = \prod_{k=1}^n \frac{1}{2i \sin \frac{\omega_k t}{2}}, \quad \text{where} \quad \hat{H}_0 = -\frac{1}{2} \hbar^2 \Delta + \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2.$$

2. *For $j \geq 1$, the wave invariants $a_j(t)$ defined in (5) are given by*

$$(7) \quad a_j(t) = a_0(t) \sum_{l=1}^{2j} i^{l(n-1) + \frac{n}{2}} e^{i \frac{\pi}{4} \text{sgn} H_l} \int_0^t \int_0^{s_1} \dots \int_0^{s_{l-1}} P_{l+j} b_l(0) ds_l \dots ds_1,$$

where for every m ,

$$P_m b_l(0) = \frac{i^{-m}}{2^m m!} \langle H_l^{-1} \nabla, \nabla \rangle^m (b_l)(0),$$

$$b_l = \prod_{i=1}^l W\left(\frac{\cos \omega_k s_i}{2} (z_{i+1}^k + z_i^k) - \frac{\sin \omega_k s_i}{\omega_k} \xi_i^k + \left(\frac{\sin \omega_k (t - s_i) + \sin \omega_k s_i}{\sin \omega_k t}\right) x^k\right),$$

and H_l^{-1} is the inverse matrix of the Hessian $H_l = \text{Hess } \Psi_l(0)$, where

$$\Psi_l = \Psi_l(t, x, z_1, \dots, z_l, \xi_1, \dots, \xi_l) = \sum_{k=1}^n \left\{ (-\omega_k \tan \frac{\omega_k}{2} t) x_k^2 + \left(\frac{\omega_k}{2} \cot \omega_k t \right) (z_1^k)^2 + \sum_{i=1}^l (z_{i+1}^k - z_i^k) \xi_i^k \right\}.$$

The Hessian of Ψ_l is calculated with respect to every variable except t . Therefore the entries of the matrix H_l^{-1} are functions of t . The matrix H_l^{-1} is shown in (34).

3. The wave invariant $a_j(t)$ is a polynomial of degree $2j$ of the Taylor coefficients of V . The highest order of derivatives appearing in $a_j(t)$ are of order $2j + 2$. In fact these higher order derivatives appear in the linear term of the polynomial and

(8)

$$a_j(t) = \frac{a_0(t)}{(2i)^{j+1}} \sum_{|\vec{\alpha}|=j+1} \frac{t}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}}{2} \right)^{\vec{\alpha}} D_{2\vec{\alpha}}^{2j+2} V(0) + \{a \text{ polynomial of Taylor coefficients of order } \leq 2j + 1\}$$

Notice that in (8), we have used the standard shorthand notations for multi-indices, i.e. $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\vec{\omega} = (\omega_1, \dots, \omega_n)$, $|\vec{\alpha}| = \alpha_1 + \dots + \alpha_n$, $\vec{\alpha}! = \alpha_1! \dots \alpha_n!$, $\vec{X}^{\vec{\alpha}} = X_1^{\alpha_1} \dots X_n^{\alpha_n}$, and $D_{\vec{\alpha}}^m = \frac{\partial^m}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ with $m = |\vec{\alpha}|$.

Our second result improves the result of Guillemin and Uribe in [GU]. This theorem is actually a non-trivial corollary of Theorem 1.1.

THEOREM 1.2. *Let V satisfy (2), (6), and be of the form*

$$(9) \quad V(x) = f(x_1^2, \dots, x_n^2) + x_n^3 g(x_1^2, \dots, x_n^2),$$

for some $f, g \in C^\infty(\mathbb{R}^n)$. Then the low-lying eigenvalues of $\hat{P} = -\frac{1}{2}\hbar^2 \Delta + V$ determine $D_{\vec{\alpha}}^{|\vec{\alpha}|} V(0)$, $|\vec{\alpha}| = 2, 3$, and if $D_{3\vec{e}_n}^3 V(0) := \frac{\partial^3 V}{\partial x_n^3}(0) \neq 0$, they determine all the Taylor coefficients of V at $x = 0$.

One quick consequence of Theorem 1.2 is the following:

COROLLARY 1.3. *If $n = 1$, and $V \in C^\infty(\mathbb{R})$ satisfies (2), then (with no symmetry assumptions) the low-lying eigenvalues determine $V''(0)$ and $V^{(3)}(0)$, and if $V^{(3)}(0) \neq 0$, then these eigenvalues determine all the Taylor coefficients of V at $x = 0$.*

Let us briefly sketch our main ideas for the proofs. First, because of a technical reason which arises in the proofs, we will need to replace the Hamiltonian P by the following Hamiltonian H

$$\begin{cases} H(x, \xi) = \frac{1}{2} \xi^2 + V_\hbar(x), \\ V_\hbar(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W_\hbar(x), \\ W_\hbar(x) = \chi\left(\frac{x}{\hbar^{\frac{1}{2}-\varepsilon}}\right) W(x), \quad \varepsilon > 0 \text{ sufficiently small,} \end{cases}$$

where the cut off $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the unit ball $B_1(0)$ and equals one in $B_{\frac{1}{2}}(0)$.

Then in two lemmas (Lemma 2.1 and Lemma 2.2) we show that for t is a sufficiently small interval $(0, t_0)$, in the sense of tempered distributions we have

$$\text{Tr}(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) = \text{Tr}(e^{\frac{-it}{\hbar}\hat{H}}) + O(\hbar^\infty).$$

This reduces the problem to studying the asymptotic of $\text{Tr}(e^{\frac{-it}{\hbar}\hat{H}})$. For this we use the construction of the kernel $k(t, x, y)$ of the propagator $U(t) = e^{\frac{-it}{\hbar}\hat{H}}$ found in [Z]. In [Z] it is shown that

$$(10) \quad k(t, x, y) = C(t) e^{\frac{i}{\hbar} S(t, x, y)} \sum_{l=0}^{\infty} a_l(t, \hbar, x, y),$$

where

$$(11) \quad S(t, x, y) = \sum_{k=1}^n \frac{\omega_k}{\sin \omega_k t} \left(\frac{1}{2} (\cos \omega_k t) (x_k^2 + y_k^2) - x_k y_k \right),$$

$a_0 = 1$, and for $l \geq 1$,

$$a_l(t, \hbar, x, y) = \left(\frac{-1}{2\pi} \right)^{ln} \left(\frac{1}{i\hbar} \right)^{l(n+1)} \int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l} e^{\frac{i}{\hbar} \Phi_l} b_l(s, x, y, \vec{z}, \vec{\xi}) d^l z d^l \xi d^l s,$$

where

$$\Phi_l = \sum_{k=1}^n \left\{ \frac{\omega_k}{2} \cot \omega_k t (z_1^k)^2 + \sum_{i=1}^l (z_{i+1}^k - z_i^k) \xi_i^k \right\},$$

and

$$b_l = \prod_{i=1}^l W_{\hbar} \left(\frac{\cos \omega_k s_i}{2} (z_{i+1}^k + z_i^k) - \frac{\sin \omega_k s_i}{\omega_k} \xi_i^k + \frac{\sin \omega_k (t - s_i)}{\sin \omega_k t} y^k + \frac{\sin \omega_k s_i}{\sin \omega_k t} x^k \right).$$

Next we apply the expression in (10) for $k(t, x, y)$ to the formula $Tr(e^{\frac{-it}{\hbar} \hat{H}}) = \int k(t, x, x) dx$. Then we obtain an infinite series of oscillatory integrals, each one corresponding to one a_l . Finally we apply the method of stationary phase to each oscillatory integral and we show that the resulting series is a valid asymptotic expansion. From the resulting asymptotic expansion we obtain the formulas (7).

Now let us compare our approach for the construction of $k(t, x, y)$ with the classical approach. In the classical approach (see for instance [DSj], [D], [R], [BPU] and [U]), one constructs a WKB approximation for the kernel $k_P(t, x, y)$ of the operator $\Theta(\hat{P})e^{\frac{-it}{\hbar} \hat{P}}$, i.e.

$$(12) \quad k_P(t, x, y) = \int e^{\frac{i}{\hbar}(\varphi_P(t, x, \eta) - y \cdot \eta)} b_P(t, x, y, \eta, \hbar) d\eta,$$

where $\varphi_P(t, x, \eta)$ satisfies the Hamilton-Jacobi equation (or eikonal equation in geometrical optics)

$$\partial_t \varphi_P(t, x, \eta) + P(x, \partial_x \varphi_P(t, x, \eta)) = 0, \quad \varphi_P|_{t=0} = x \cdot \eta,$$

and the function b_P has an asymptotic expansion of the form

$$b_P(t, x, y, \eta, \hbar) \sim \sum_{j=0}^{\infty} b_{P,j}(t, x, y, \eta) \hbar^j.$$

The functions $b_{P,j}(t, x, y, \eta)$ are calculated from the so called transport equations. See for example [R], [DSj], [EZ] or Appendix A of the paper in hand for the details of the above construction.

In this setting, when one integrates the kernel $k(t, x, y)$ on the diagonal and applies the stationary phase to the given oscillatory integral, one obtains very complicated expressions for the wave invariants. Of course the classical calculations above show the existence of asymptotic formulas of the form (5) (which can be used to get Weyl-type estimates for the counting functions of the eigenvalues, see for example [BPU]). Unfortunately these formulas for the wave invariants are not helpful when trying to establish some inverse spectral results.

Hence, one should look for more efficient methods to calculate the wave invariants $a_j(t)$. One approach is to use the semi-classical Birkhoff normal forms, which was used in [GU], to get some inverse spectral results as mentioned in the beginning of the introduction. The Birkhoff normal forms methods were also used before by S. Zelditch in [Z4] to obtain positive inverse spectral results for real analytic domains with symmetries of an ellipse. Zelditch proved that for a real analytic plane domain with symmetries of an ellipse, the wave invariants at a bouncing ball orbit, which is preserved by the symmetries, determine the real analytic domain under isometries of the domain.

Recently in [Z3], Zelditch improved his earlier result to real analytic domains with only one mirror symmetry. His approach for this new result was different. He used a direct approach (Balian-Bloch trace formula) which involves Feynman-diagrammatic calculations of the stationary phase method to obtain a more explicit formula for the wave invariants at the bouncing ball orbit.

Motivated by the work of Zelditch [Z3] mentioned above, our approach in this article is also somehow direct and involves combinatorial calculations of the stationary phase.

Our formula in (10) for the kernel of the propagator, $U(t) = e^{\frac{-it}{\hbar} \hat{H}}$, is different from the WKB-expression in the sense that we only keep the quadratic part of the phase function, namely the phase function $S(t, x, y)$ in (11) of the propagator of Anisotropic oscillator, and we put the rest in the amplitude $\sum_{l=0}^{\infty} a_l(t, \hbar, x, y)$

in (10). The details of this construction are mentioned in Section 2.2.

REMARK 1.4. *After the initial posting of this article, Guillemin and Colin de Verdière posted two articles (see [CG1], also [C]) in which they study some inverse spectral problems of 1 dimensional semi-classical Schrödinger operators. One of the main results in [CG1] is our Corollary 1.3 in this paper.*

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2. PROOFS OF THE RESULTS

2.1. Some reductions. Because of some technical issues arising in the proof of Theorem 1.1, we will need to use the following lemmas as reductions.

In the following, we let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut off which is supported in the unit ball $B_1(0)$ and equals one in $B_{\frac{1}{2}}(0)$.

LEMMA 2.1. *Let the Hamiltonians P and H be defined by*

$$(13) \quad \left\{ \begin{array}{l} P(x, \xi) = \frac{1}{2} \xi^2 + V(x) \\ V(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W(x) \end{array} \right., \quad \left\{ \begin{array}{l} H(x, \xi) = \frac{1}{2} \xi^2 + V_h(x), \\ V_h(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W_h(x), \\ W_h(x) = \chi\left(\frac{x}{h^{\frac{1}{2}-\varepsilon}}\right)W(x), \quad \varepsilon > 0 \text{ sufficiently small,} \end{array} \right.$$

and let \hat{P} and \hat{H} be the corresponding Weyl (or standard) quantizations. Then for t in a sufficiently small interval $(0, t_0)$

$$\text{Tr}(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) = \text{Tr}(\Theta(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}) + O(\hbar^\infty).$$

In other words, the wave invariants $a_j(t)$ will not change if we replace P by H .

PROOF. Proof is given in Appendix A.

Next we use the following lemma to get rid of $\Theta(\hat{H})$.

LEMMA 2.2. *Let H be defined by (13). Then in the sense of tempered distributions*

$$\text{Tr}(\Theta(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}) = \text{Tr}(e^{\frac{-it}{\hbar}\hat{H}}) + O(\hbar^\infty).$$

This means that if we sort the spectrum of \hat{H} as

$$E_1(\hbar) < E_2(\hbar) \leq \dots \leq E_j(\hbar) \rightarrow +\infty,$$

then for every Schwartz function $\varphi(t) \in S(\mathbb{R})$

$$\langle \text{Tr}(e^{\frac{-it}{\hbar}\hat{H}}) - \text{Tr}(\Theta(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}), \varphi(t) \rangle = \sum_{j=1}^{\infty} (1 - \Theta(E_j(\hbar))) \hat{\varphi}\left(\frac{E_j(\hbar)}{\hbar}\right) = O(\hbar^\infty).$$

PROOF. Proof is given in Appendix B.

Because of the above lemmas, it is enough to study the asymptotic of $\text{Tr}(e^{\frac{-it}{\hbar}\hat{H}})$.

2.2. Construction of $k(t, x, y)$, the kernel of $e^{\frac{-it}{\hbar}\hat{H}}$. In this section we follow the construction in [Z] to obtain an oscillatory integral representation of $k(t, x, y)$, the kernel of the propagator $e^{\frac{-it}{\hbar}\hat{H}}$. The reader should consult [Z] for many details. In that article Zelditch uses the Dyson's Expansion of propagator to study the singularities of the kernel $k(t, x, y)$. But he does not consider the semi-classical setting $\hbar \rightarrow 0$ in his calculations (i.e. $\hbar = 1$). So we follow the same calculations but also consider \hbar carefully.

In [Z], potentials of the form

$$V(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W_{\hbar}(x),$$

are considered, where $W_{\hbar} \in B(\mathbb{R}^n)$, i.e. bounded with bounded derivatives.

We denote

$$\begin{cases} \hat{H}_0 = -\frac{1}{2}\hbar^2\Delta + \frac{1}{2}\sum_{k=1}^n \omega_k^2 x_k^2, & (\text{Anisotropic Oscillator}) \\ \hat{H} = \hat{H}_0 + W_{\hbar}(x) = -\frac{1}{2}\hbar^2\Delta + V_{\hbar}(x), \end{cases}$$

and by $U_0(t) = e^{\frac{-it}{\hbar}\hat{H}_0}$, and $U(t) = e^{\frac{-it}{\hbar}\hat{H}}$, we mean their corresponding propagators.

From

$$(i\hbar\partial_t - \hat{H}_0)U(t) = W_{\hbar}.U(t),$$

we obtain

$$(14) \quad U(t) = U_0(t) + \frac{1}{i\hbar} \int_0^t U_0(t-s).W_{\hbar}.U(s)ds.$$

By iteration we get the norm convergent Dyson Expansion:

$$(15) \quad U(t) = U_0(t) + \sum_{l=1}^{\infty} \frac{1}{(i\hbar)^l} \int_0^t \dots \int_0^{s_{l-1}} U_0(t)[U_0(s_1)^{-1}.W_{\hbar}.U_0(s_1)] \dots [U_0(s_l)^{-1}.W_{\hbar}.U_0(s_l)]ds_l \dots ds_1.$$

For $t \neq \frac{m\pi}{\omega_k}$, the kernel of $U_0(t)$ is given by

$$(16) \quad k_0(t, x, y) = \left(\prod_{k=1}^n \frac{\omega_k}{2\pi i \hbar \sin \omega_k t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} S(t, x, y)},$$

where

$$S(t, x, y) = \sum_{k=1}^n \frac{\omega_k}{\sin \omega_k t} \left(\frac{1}{2} (\cos \omega_k t) (x_k^2 + y_k^2) - x_k y_k \right).$$

Then by taking kernels in (15) and after some change of variables (see [Z]), we get

$$(17) \quad k(t, x, y) = \left(\prod_{k=1}^n \frac{\omega_k}{2\pi i \hbar \sin \omega_k t} \right)^{\frac{1}{2}} e^{\frac{i}{\hbar} S(t, x, y)} \sum_{l=0}^{\infty} a_l(t, \hbar, x, y),$$

where $a_0 = 1$ and for $l \geq 1$,

$$(18) \quad a_l(t, \hbar, x, y) = \left(\frac{-1}{2\pi} \right)^{ln} \left(\frac{1}{i\hbar} \right)^{l(n+1)} \int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l} e^{\frac{i}{\hbar} \Phi_l} b_l(s, x, y, \vec{z}, \vec{\xi}) d^l z d^l \xi d^l s,$$

where

$$(19) \quad \Phi_l = \sum_{k=1}^n \left\{ \frac{\omega_k}{2} \cot \omega_k t (z_1^k)^2 + \sum_{i=1}^l (z_{i+1}^k - z_i^k) \xi_i^k \right\},$$

and

$$(20) \quad b_l = \prod_{i=1}^l W_h \left(\frac{\cos \omega_k s_i}{2} (z_{i+1}^k + z_i^k) - \frac{\sin \omega_k s_i}{\omega_k} \xi_i^k + \frac{\sin \omega_k (t - s_i)}{\sin \omega_k t} y^k + \frac{\sin \omega_k s_i}{\sin \omega_k t} x^k \right). \quad (z_{l+1} := 0)$$

In [Z], integration by parts in the variables z, ξ , for the integrals in (18) are performed (in that article, there is no \hbar , i.e. $\hbar = 1$, because of different motivations) to prove that the sum $a(t, 1, x, y) = \sum_{l=0}^\infty a_l(t, 1, x, y)$ is absolutely uniformly convergent. Additionally, it is shown that $a(t, 1, x, y)$ is in $B(\mathbb{R}_x^n \times \mathbb{R}_y^n)$. More precisely, there exists $k_0 = k_0(\alpha, \beta, n)$ such that

$$(21) \quad |\partial_x^{\vec{\alpha}} \partial_y^{\vec{\beta}} a_l(t, 1, x, y)| \leq \frac{1}{l!} C_{\alpha, \beta, n}(t) \|W_1\|_{|\alpha|+|\beta|+k_0}^l, \quad (W_1 = W_h|_{\hbar=1})$$

which implies

$$|\partial_x^{\vec{\alpha}} \partial_y^{\vec{\beta}} a(t, 1, x, y)| \leq \exp\{C_{\alpha, \beta, n}(t) \|W_1\|_{|\alpha|+|\beta|+k_0}\}.$$

The estimates (21) will change if one considers \hbar in the calculations. We will establish these \hbar estimates in Lemma 2.3. As a simple consequence of Lemma 2.3, let us assume for now that $a(t, \hbar, x, y) \in B(\mathbb{R}_x^n \times \mathbb{R}_y^n)$.

2.3. Oscillatory Integral Representation for the Trace of $U(t) = e^{-\frac{it}{\hbar} \hat{H}}$. In this section we show that the integral $Tr U(t) = \int k(t, x, x) dx$ is convergent as an oscillatory integral and that from (17) and (18) we can write

$$(22) \quad Tr U(t) = \left(\prod_{k=1}^n \frac{\omega_k}{2\pi i \hbar \sin \omega_k t} \right)^{\frac{1}{2}} \sum_{l=0}^{\infty} \left(\frac{-1}{2\pi} \right)^{ln} \left(\frac{1}{i\hbar} \right)^{l(n+1)} \int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l+1} e^{\frac{i}{\hbar} \Psi_l} b_l(s, x, x, \vec{z}, \vec{\xi}) d^l s d^l z d^l \xi dx,$$

where

$$(23) \quad \Psi_l = S(t, x, x) + \Phi_l = \sum_{k=1}^n \left\{ (-\omega_k \tan \frac{\omega_k t}{2}) x_k^2 + \frac{\omega_k}{2} \cot \omega_k t (z_1^k)^2 + \sum_{i=1}^l (z_{i+1}^k - z_i^k) \xi_i^k \right\}.$$

Before proving (22), we review some standard facts. First of all we know that the sum

$$Tr U(t) = \sum e^{-\frac{it E_j(\hbar)}{\hbar}}$$

is convergent in the sense of tempered distributions, i.e. $Tr U(t) \in S'(\mathbb{R})$. This can be shown by the Weyl's law in its high energy setting, which implies that for potentials of the form $V(x) = \frac{1}{2} \sum_{k=1}^n \omega_k^2 x_k^2 + W_h(x)$, with $W \in B(\mathbb{R}^n)$, for fixed \hbar , the j -th eigenvalue $E_j(\hbar)$ satisfies

$$(24) \quad E_j(\hbar) \sim C(n, \hbar) j^{\frac{2}{n}}, \quad j \rightarrow \infty.$$

Another way to define $Tr U(t)$ is to write it as the limit

$$(25) \quad Tr U(t) = \lim_{\varepsilon \rightarrow 0^+} Tr U(t - i\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} \sum e^{-\frac{(it+\varepsilon) E_j(\hbar)}{\hbar}}.$$

This time the Weyl's law (24) implies that the sum $Tr U(t - i\varepsilon)$ is absolutely uniformly convergent because of the rapidly decaying factor $e^{-\frac{\varepsilon E_j(\hbar)}{\hbar}}$. As a result, $U(t - i\varepsilon)$ is a trace class operator. It is clear that the kernel of $U(t - i\varepsilon)$ is $k(t - i\varepsilon, x, y)$, the analytic continuation of the kernel $k(t, x, y)$ of $U(t)$. Clearly $k(t - i\varepsilon, x, y)$ is continuous in x and y . So we can write $Tr U(t - i\varepsilon) = \int k(t - i\varepsilon, x, x) dx$. We notice that this integral is uniformly convergent. This is because up to a constant this integral equals to $\int e^{\frac{i}{\hbar} S(t - i\varepsilon, x, x)} a(t - i\varepsilon, \hbar, x, x)$, and the exponential factor in the integral is rapidly decaying for $\varepsilon > 0$ as $|x| \rightarrow \infty$ and a is a bounded function. More precisely

$$\Re(iS(t - i\varepsilon, x, x)) = \sum_{k=1}^n \Re(-i\omega_k \tan(\frac{\omega_k(t - i\varepsilon)}{2})) x_k^2 = \sum_{k=1}^n \frac{\omega_k(1 - e^{2\varepsilon\omega_k})}{|1 + e^{w_k(it+\varepsilon)}|^2} x_k^2,$$

and

$$\frac{\omega_k(1 - e^{2\varepsilon\omega_k})}{|1 + e^{w_k(it+\varepsilon)}|^2} < 0.$$

The discussion above shows that the integral $\int k(t, x, x)dx$ can be defined by integrations by parts as follows: Since

$$\langle D_x \rangle^2 e^{iS(t, x, x)} := (1 - \Delta)e^{iS(t, x, x)} = (1 + \|2\vec{\omega} \tan(\frac{\vec{\omega}t}{2})\vec{x}\|^2 + 2i \sum_{k=1}^n \omega_k \tan(\frac{\omega_k t}{2}))e^{iS(t, x, x)},$$

we can write

$$(26) \quad \int e^{\frac{i}{\hbar}S(t, x, x)} a(t, \hbar, x, x) dx = \hbar^{\frac{n}{2}} \int e^{iS(t, x, x)} a(t, \hbar, \sqrt{\hbar}x, \sqrt{\hbar}x) dx = \\ \hbar^{\frac{n}{2}} \int e^{iS(t, x, x)} (\langle D_x \rangle^2 (1 + \|2\vec{\omega} \tan(\frac{\vec{\omega}t}{2})\vec{x}\|^2 + 2i \sum_{k=1}^n \omega_k \tan(\frac{\omega_k t}{2}))^{-1})^{n_0} a(t, \hbar, \sqrt{\hbar}x, \sqrt{\hbar}x) dx$$

If we assume $0 < t < \min_{1 \leq k \leq n} \{\frac{\pi}{2\omega_k}\}$, then by choosing $n_0 > \frac{n}{2}$, and because $a(t, \hbar, x, y) \in B(\mathbb{R}_x^n \times \mathbb{R}_y^n)$, the integral becomes absolutely convergent. Finally, since by (27) the series $a(t, \hbar, x, y) = \sum_{l=0}^{\infty} a_l(t, \hbar, x, y)$ is absolutely uniformly convergent, we have

$$\int e^{\frac{i}{\hbar}S(t, x, x)} a(t, \hbar, x, x) dx = \sum_{l=0}^{\infty} \int e^{\frac{i}{\hbar}S(t, x, x)} a_l(t, \hbar, x, x) dx,$$

and therefore we obtain (22), which is an infinite sum of oscillatory integrals. The next step is to apply the stationary phase method to each integral in (22) and then add the asymptotics to obtain an asymptotic expansion for the $Tr U(t)$. Because we have an infinite sum of asymptotic expansions, we have to establish that the resulting asymptotic for the trace is a valid approximation. Hence we have to find some appropriate \hbar -estimates for the remainder term of the series (22).

2.4. \hbar -estimates for the remainder. The goal of this section is to find an \hbar -estimate for the remainder term of the series (22). First we state the following important estimate, which shows how the estimates in (21) change in the \hbar -dependence case.

LEMMA 2.3. *For every $\vec{\alpha}, \vec{\beta}$, there exists $k_0 = k_0(\alpha, \beta)$ such that for every $0 < \hbar \leq h_0 \leq 1$*

$$(27) \quad |\partial_x^{\vec{\alpha}} \partial_y^{\vec{\beta}} a_l(t, \hbar, x, y)| \leq \frac{C_{\alpha, \beta, n}(t) \|W_{\hbar}^*\|^l}{l!} \hbar^{l(\frac{1}{2} - 3\varepsilon)},$$

where

$$(28) \quad W_{\hbar}^*(x) = \frac{W_{\hbar}(\hbar^{\frac{1}{2}}x)}{\hbar^{3(\frac{1}{2} - \varepsilon)}} = \chi(\hbar^{\varepsilon}x) \frac{W(\hbar^{\frac{1}{2}}x)}{\hbar^{3(\frac{1}{2} - \varepsilon)}}$$

is uniformly in $B(\mathbb{R}_x^n)$; i.e. W_{\hbar}^* is bounded with bounded derivatives and the bounds are independent of \hbar .

Before proving this lemma, let us show how the lemma is applied to get estimates for the remainder term of the series (22). Define $I_l(t, \hbar)$ to be the l -th term in (22) (removing the constant $\prod_{k=1}^n (\frac{\omega_k}{2\pi i \sin \omega_k t})^{\frac{1}{2}}$), i.e.

$$(29) \quad I_l(t, \hbar) = \hbar^{-\frac{n}{2}} \int e^{\frac{i}{\hbar}S(t, x, x)} a_l(t, \hbar, x, x) dx.$$

Hence by this notation, $Tr U(t) = \prod_{k=1}^n (\frac{\omega_k}{2\pi i \sin \omega_k t})^{\frac{1}{2}} \sum_{l=0}^{\infty} I_l(t, \hbar)$. If in (29) we integrate by parts as we did in (26), and choose $n_0 = [\frac{n}{2}] + 1$, then using (27) we get

$$|I_l(t, \hbar)| \leq C'_n(t) \frac{C_n(t) \|W_{\hbar}^*\|^l}{l!} \hbar^{l(\frac{1}{2} - 3\varepsilon)},$$

where $C_n(t) = \max_{|\alpha|+|\beta| \leq 2n_0} \{C_{\alpha, \beta, n}(t)\}$. We choose $\varepsilon > 0$ such that $\frac{1}{2} - 3\varepsilon > 0$, or $\varepsilon < \frac{1}{6}$. Now it is clear that for every positive integer m , and every $0 < \hbar \leq h_0 \leq 1$,

$$(30) \quad \left| \sum_{l=m}^{\infty} I_l(t, \hbar) \right| \leq C'_n(t) e^{\{C_n(t) \|W_{\hbar}^*\|_{2n_0+k_0}\}} \hbar^{m(\frac{1}{2}-3\varepsilon)}.$$

Since by Lemma 2.3, $\sup_{0 < \hbar \leq 1} \|W_{\hbar}^*\|_{2n_0+k_0} < \infty$, we have

$$(31) \quad \text{Tr } U(t) = \prod_{k=1}^n \left(\frac{\omega_k}{2\pi i \sin \omega_k t} \right)^{\frac{1}{2}} \sum_{l=0}^{m-1} I_l(t, \hbar) + O(\hbar^{m(\frac{1}{2}-3\varepsilon)}).$$

This is a useful estimate that we will use in the next section to obtain the asymptotic expansion of the trace and prove Theorem 1.1. Let us first prove Lemma 2.3.

PROOF OF LEMMA 2.3:

The proof is straightforward from (21). First, in (18), we apply the change of variables $\vec{z} \mapsto \hbar^{\frac{1}{2}} \vec{z}$ and $\vec{\xi} \mapsto \hbar^{\frac{1}{2}} \vec{\xi}$. This gives us \hbar^{ln} in front of the integral. Then we replace W_{\hbar} by $\hbar^{3(\frac{1}{2}-\varepsilon)} W_{\hbar}^*$. After collecting all the powers of \hbar in front of the integral we obtain

$$a_l(t, \hbar, x, y) = \left(\frac{-1}{2\pi} \right)^{ln} \hbar^{l(\frac{1}{2}-3\varepsilon)} \int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l} e^{i\Phi_l} b_l^*(s, x, y, \vec{z}, \vec{\xi}) d^l z d^l \xi d^l s,$$

where

$$b_l^*(s, x, y, \vec{z}, \vec{\xi}) = \prod_{i=1}^l W_{\hbar}^* \left(\frac{\cos \omega_k s_i}{2} (z_{i+1}^k + z_i^k) - \frac{\sin \omega_k s_i}{\omega_k} \xi_i^k + \frac{\sin \omega_k (t - s_i)}{\sin \omega_k t} y^k + \frac{\sin \omega_k s_i}{\sin \omega_k t} x^k \right).$$

Next we apply (21) to the above integral with W_1 replaced by W_{\hbar}^* , and we get (27). To finish the proof we have to show that for every positive integer m we can find uniform bounds (i. e. independent of \hbar) for the m -th derivatives of the function $W_{\hbar}^*(x)$. Since $\chi(x)$ is supported in the unit ball, from the definition (28) we see that W_{\hbar}^* is supported in $|x| < \hbar^{-\varepsilon}$. So from (28) it is enough to find uniform bounds in \hbar for the m -th derivatives of the function $\frac{W(\hbar^{\frac{1}{2}} x)}{\hbar^{3(\frac{1}{2}-\varepsilon)}}$ in the ball $|x| < \hbar^{-\varepsilon}$. This is very clear for $m \geq 3$. For $m < 3$, we use the order of vanishing of $W(x)$ at $x = 0$. Since $W(0) = \nabla W(0) = \text{Hess} W(0) = 0$, the order of vanishing of W at $x = 0$ is 3. Therefore in the ball $|x| < \hbar^{-\varepsilon}$, the functions

$$\frac{W(\hbar^{\frac{1}{2}} x)}{(\hbar^{\frac{1}{2}} x)^3}, \frac{\partial^{\alpha} W(\hbar^{\frac{1}{2}} x)}{(\hbar^{\frac{1}{2}} x)^2}, \frac{\partial^{\alpha} \partial^{\beta} W(\hbar^{\frac{1}{2}} x)}{\hbar^{\frac{1}{2}} x},$$

are bounded functions with uniform bounds in \hbar , and the statement follows easily for $m < 3$, noting that $|x| < \hbar^{-\varepsilon}$.

2.5. Stationary phase calculations. In this section we will apply the stationary phase method to each $I_l(t, \hbar)$ in (31). We know

$$(32) \quad I_l(x, \hbar) = \left(\frac{-1}{2\pi} \right)^{ln} \left(\frac{1}{i\hbar} \right)^{l(n+1)+\frac{n}{2}} \int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l+1} e^{\frac{i}{\hbar} \Psi_l} b_l(s, x, x, \vec{z}, \vec{\xi}) d^l s d^l z d^l \xi dx.$$

It is easy to see that the only critical point of the phase function Ψ_l , given by (23), is at $(x, \vec{z}, \vec{\xi}) = 0$.

Next we calculate $H_l = \text{Hess}\Psi_l(0)$ and H_l^{-1} . In the following we use the notation $D(\vec{v})$ for the diagonal matrix $\text{Diag}(v_1, \dots, v_n)$, where $\vec{v} = (v_1, \dots, v_n)$. From (23), we get

$$(33) \quad H_l = \begin{pmatrix} D(-2\vec{\omega} \tan(\frac{\vec{\omega}t}{2}))_{n \times n} & 0 & 0 \\ 0 & \begin{pmatrix} D(\vec{\omega} \cot \vec{\omega}t)_{n \times n} & 0 \\ 0 & 0 \end{pmatrix}_{ln \times ln} & \begin{pmatrix} -I & 0 & \dots & 0 \\ I & -I & 0 & \dots & 0 \\ 0 & I & \ddots & & \\ \vdots & & \ddots & & \\ 0 & & & I & -I \end{pmatrix}_{ln \times ln} \\ 0 & \begin{pmatrix} -I & I & \dots & 0 \\ 0 & -I & I & \dots & 0 \\ \cdot & 0 & \ddots & \ddots & \\ \cdot & & & I & \\ 0 & & & 0 & -I \end{pmatrix}_{ln \times ln} & 0 \end{pmatrix}_{(2l+1)n \times (2l+1)n},$$

where $I = I_{n \times n}$ is the identity matrix of size $n \times n$.

Since H_l is of the form $H_l = \begin{pmatrix} K & 0 & 0 \\ 0 & A & B \\ 0 & B^T & 0 \end{pmatrix}$, the inverse matrix equals $H_l^{-1} = \begin{pmatrix} K^{-1} & 0 & 0 \\ 0 & 0 & B^{T^{-1}} \\ 0 & B^{-1} & -B^{-1}AB^{T^{-1}} \end{pmatrix}$.

A simple calculation shows that

$$(34) \quad H_l^{-1} = \begin{pmatrix} D(\frac{-1}{2\vec{\omega}} \cot(\frac{\vec{\omega}t}{2})) & 0 & 0 \\ 0 & 0 & \begin{pmatrix} -I & -I & \dots & -I \\ 0 & -I & -I & \dots & -I \\ 0 & 0 & -I & \ddots & \vdots \\ \vdots & \ddots & \ddots & & -I \\ 0 & \dots & & 0 & -I \end{pmatrix} \\ 0 & \begin{pmatrix} -I & 0 & \dots & 0 \\ -I & -I & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -I & \ddots & \ddots & -I & 0 \\ -I & \dots & & -I & -I \end{pmatrix} & \begin{pmatrix} -\Omega & -\Omega & \dots & -\Omega \\ -\Omega & -\Omega & \dots & -\Omega \\ \vdots & & & \vdots \\ -\Omega & -\Omega & \dots & -\Omega \end{pmatrix} \end{pmatrix},$$

where $\Omega = D(\vec{\omega} \cot \vec{\omega}t)$.

It is also easy to see that

$$(35) \quad \det H_l = (-1)^{(l+1)n} \prod_{k=1}^n 2\omega_k \tan \frac{\omega_k t}{2}.$$

By applying the stationary phase lemma to (32) and plugging into (31) we obtain

$$(36) \quad \text{Tr} U(t) = \prod_{k=1}^n \left(\frac{\omega_k}{\sin \omega_k t} \right)^{\frac{1}{2}} \sum_{l=0}^{m-1} \frac{\hbar^{-l}}{i^{l(n+1)+\frac{n}{2}}} \frac{(-1)^{ln} e^{i\frac{\pi}{4} \text{sgn} H_l}}{\sqrt{|\det H_l|}} \int_0^t \dots \int_0^{s_{l-1}} \sum_{j=0}^{\infty} \hbar^j P_j b_l(0) ds_l \dots ds_1 + O(\hbar^{m(\frac{1}{2}-3\varepsilon)}),$$

where

$$(37) \quad P_j b_l(x, \vec{z}, \vec{\xi}) = \frac{i^{-j}}{2^j \cdot j!} \langle H_l^{-1} \nabla, \nabla \rangle^j b_l(x, \vec{z}, \vec{\xi}) = \frac{i^{-j}}{2^j \cdot j!} \sum_{r_1, \dots, r_{2j} \in \mathcal{A}_l} h_l^{r_1 r_2} \dots h_l^{r_{2j-1} r_{2j}} \frac{\partial^{2j} b_l(x, \vec{z}, \vec{\xi})}{\partial r_1 \dots \partial r_{2j}},$$

where in the sum (37) the indices r_1, \dots, r_{2j} run in the set $\mathcal{A}_l = \{x^k, z_1^k, \dots, z_l^k, \xi_1^k, \dots, \xi_l^k\}_{k=1}^n$, and $h_l^{rr'}$ with $r, r' \in \mathcal{A}_l$, corresponds to the (r, r') -th entry of the inverse Hessian H_l^{-1} .

We note that $P_j b_l(0) = 0$ if $2j < 3l$. This is true because of (20) and because $W(0) = \nabla W(0) = \text{Hess}W(0) = 0$. This implies, first, there are not any negative powers of \hbar in the expansion (as we were expecting). Second, the constant term (i.e. the 0-th wave invariant), which corresponds to the term $l = j = 0$ in the sum, equals

$$a_0(t) = \text{Tr} U_0(t) = \prod_{k=1}^n \left(\frac{\omega_k}{\sin \omega_k t} \right)^{\frac{1}{2}} \frac{i^{-\frac{n}{2}} e^{\frac{-\pi n}{4}}}{(2\omega_k \tan \frac{\omega_k t}{2})^{\frac{1}{2}}} = \prod_{k=1}^n \frac{1}{2i \sin \frac{\omega_k t}{2}}.$$

And third (using (35)), for $j \geq 1$ the coefficient of \hbar^j in (36) equals

$$a_j(t) = \left(\prod_{k=1}^n \frac{1}{2i \sin \frac{\omega_k t}{2}} \right) \sum_{l=1}^{2j} i^{l(n-1) + \frac{n}{2}} e^{i \frac{\pi}{4} \text{sgn} H_l} \int_0^t \int_0^{s_1} \dots \int_0^{s_{l-1}} P_{l+j} b_l(0) ds_l \dots ds_1.$$

The sum goes only up to $2j$ because if $l > 2j$ then $2(l+j) < 3l$ and $P_{l+j} b_l(0) = 0$.

This proves the first two parts of Theorem 1.1.

2.6. Calculations of the wave invariants and the proof of Theorem 1.1.3. In this section we try to calculate the wave invariants $a_j(t)$ from the formulas (7). First of all, let us investigate how the terms with highest order of derivatives appear in $a_j(t)$. Because b_l is the product of l copies of W_{\hbar} functions, and because we have to put at least 3 derivatives on each W_{\hbar} to obtain non-zero terms, the highest possible order of derivatives that can appear in $P_{j+l} b_l(0)$, is $2(j+l) - 3(l-1) = 2j - l + 3$. This implies that, because in the sum (7) we have $1 \leq l \leq 2j$, the highest order of derivatives in $a_j(t)$ is $2j + 2$ and those derivatives are produced by the term corresponding to $l = 1$, i.e. $P_{j+1} b_1(0)$. The formula (7) also shows that $a_j(t)$ is a polynomial of degree $2j$. The term with the highest polynomial order is the one with $l = 2j$, i.e. $P_{3j} b_{2j}(0)$ (which has the lowest order of derivatives) and the term $P_{j+1} b_1(0)$ is the linear term of the polynomial. Now let us calculate $P_{j+1} b_1(x, \vec{z}, \vec{\xi})$ and prove Theorem 1.1.3.

By (37),

$$(38) \quad P_{j+1} b_1 = \frac{i^{-(j+1)}}{2^{j+1} \cdot (j+1)!} \sum_{r_1, \dots, r_{2j+2} \in \mathcal{A}_1} h_1^{r_1 r_2} \dots h_1^{r_{2j+1} r_{2j+2}} \frac{\partial^{2j+2} b_1}{\partial r_1 \dots \partial r_{2j+2}},$$

where here by (20)

$$b_1 = W_{\hbar} \left(\frac{\cos \omega_k s}{2} z^k - \frac{\sin \omega_k s}{\omega_k} \xi^k + \left(\frac{\sin \omega_k (t-s) + \sin \omega_k s}{\sin \omega_k t} \right) x^k \right).$$

Also by (34),

$$H_1^{-1} = \begin{pmatrix} D(\frac{-1}{2\vec{\omega}} \cot(\frac{\vec{\omega} t}{2})) & 0 & 0 \\ 0 & 0 & -I \\ 0 & -I & -D(\vec{\omega} \cot \vec{\omega} t) \end{pmatrix}.$$

Hence the only non-zero entries of H_1^{-1} are the ones of the form $h_1^{x^k x^k}$, $h_1^{z^k \xi^k} = h_1^{\xi^k z^k}$, and $h_1^{\xi^k \xi^k}$. Now we let

$$\begin{cases} i_{x^k x^k} = \text{the number of times } h_1^{x^k x^k} \text{ appears in } h_1^{r_1 r_2} \dots h_1^{r_{2j+1} r_{2j+2}} \text{ in (38),} \\ i_{z^k \xi^k} = \text{the number of times } h_1^{z^k \xi^k} \text{ appears in } h_1^{r_1 r_2} \dots h_1^{r_{2j+1} r_{2j+2}} \text{ in (38),} \\ i_{\xi^k \xi^k} = \text{the number of times } h_1^{\xi^k \xi^k} \text{ appears in } h_1^{r_1 r_2} \dots h_1^{r_{2j+1} r_{2j+2}} \text{ in (38).} \end{cases}$$

By applying these notations to (38), (20) we get

$$P_{j+1}b_1 = \frac{i^{-(j+1)}}{2^{j+1}(j+1)!} \sum_{\sum_{k=1}^n i_{x^k x^k} + i_{z^k \xi^k} + i_{\xi^k \xi^k} = j+1} \frac{(j+1)! 2^{\sum_{k=1}^n i_{z^k \xi^k}}}{\prod_{k=1}^n i_{x^k x^k}! i_{z^k \xi^k}! i_{\xi^k \xi^k}!} \prod_{k=1}^n \left(\frac{-\cot \frac{\omega_k t}{2}}{2\omega_k} \right)^{i_{x^k x^k}} (-1)^{i_{z^k \xi^k}} (-\omega_k \cot \omega_k t)^{i_{\xi^k \xi^k}} \\ \times \prod_{k=1}^n \left\{ \left(\frac{\sin \omega_k(t-s) + \sin \omega_k s}{\sin \omega_k t} \right)^{2i_{x^k x^k}} \left(\frac{\cos \omega_k s}{2} \right)^{i_{z^k \xi^k}} \left(\frac{-\sin \omega_k s}{\omega_k} \right)^{i_{z^k \xi^k} + 2i_{\xi^k \xi^k}} \right\} . D_{2\alpha_1, \dots, 2\alpha_n}^{2j+2} W_h,$$

where $\alpha_k = i_{x^k x^k} + i_{z^k \xi^k} + i_{\xi^k \xi^k}$, for $k = 1, \dots, n$.

Next we write the above big sum as

$$\sum_{\sum_{k=1}^n i_{x^k x^k} + i_{z^k \xi^k} + i_{\xi^k \xi^k} = j+1} \frac{(j+1)!}{\prod_{k=1}^n i_{x^k x^k}! i_{z^k \xi^k}! i_{\xi^k \xi^k}!} (\star) = \sum_{\sum \alpha_k = j+1} \frac{(j+1)!}{\prod \alpha_k!} \sum_{i_{x^k x^k} + i_{z^k \xi^k} + i_{\xi^k \xi^k} = \alpha_k} \prod_{k=1}^n \frac{\alpha_k!}{i_{x^k x^k}! i_{z^k \xi^k}! i_{\xi^k \xi^k}!} (\star).$$

So the coefficient of $D_{2\alpha_1, \dots, 2\alpha_n}^{2j+2} W_h$ in $P_{j+1}b_1$, equals

$$\frac{i^{-(j+1)}}{2^{j+1}} \frac{(-1)^{j+1}}{(\prod \alpha_k!) (\prod \omega_k^{\alpha_k})} \prod_{k=1}^n \left(\frac{1}{2} \cot \frac{\omega_k t}{2} \left(\frac{\sin \omega_k(t-s) + \sin \omega_k s}{\sin \omega_k t} \right)^2 - \cos \omega_k s \sin \omega_k s + \cot \omega_k t \sin^2 \omega_k s \right)^{\alpha_k}.$$

Now we observe that the term in the parenthesis simplifies to

$$\frac{1}{2} \cot \frac{\omega_k t}{2} \left(\frac{\sin \omega_k(t-s) + \sin \omega_k s}{\sin \omega_k t} \right)^2 - \cos \omega_k s \sin \omega_k s + \cot \omega_k t \sin^2 \omega_k s = \frac{1}{2} \cot \frac{\omega_k t}{2}.$$

So we get

$$(39) \quad P_{j+1}b_1 = \frac{1}{(2i)^{j+1}} \sum_{|\vec{\alpha}|=j+1} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega} t}{2} \right)^{\vec{\alpha}} D_{2\vec{\alpha}}^{2j+2} W_h,$$

Finally, by plugging $(x, \vec{z}, \vec{\xi}) = 0$ into equation (39) and applying it to (7), we get (8). This finishes the proof of Theorem 1.1.3.

For future reference let us highlight the equation we just proved

$$(40) \quad S_1 := \sum_{r_1, \dots, r_{2j+2} \in \mathcal{A}_1} h_1^{r_1 r_2} \dots h_1^{r_{2j+1} r_{2j+2}} \frac{\partial^{2j+2} W}{\partial r_1 \dots \partial r_{2j+2}} = (j+1)! \sum_{|\vec{\alpha}|=j+1} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega} t}{2} \right)^{\vec{\alpha}} D_{2\vec{\alpha}}^{2j+2} W,$$

where

$$W = W\left(\frac{\cos \omega_k s}{2} z^k - \frac{\sin \omega_k s}{\omega_k} \xi^k + \left(\frac{\sin \omega_k(t-s) + \sin \omega_k s}{\sin \omega_k t}\right) x^k\right).$$

2.7. Calculations of $\int_0^t \int_0^{s_1} P_{j+2}b_2(0)$, and the proof of Theorem 1.2. Throughout this section we assume that V is of the form (9). Hence, the only non-zero Taylor coefficients are of the form $D_{2\vec{\alpha}}^{2j+2} V(0)$, or $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0)$, where $\vec{e}_n = (0, \dots, 0, 1)$.

We notice that based on our discussion in the previous section, the Taylor coefficients of order $2j+1$ appear in $\int_0^t \int_0^{s_1} P_{j+2}b_2(0)$, and they are of the form $D_{\vec{\beta}}^{2j+1} V(0) D_{\vec{\delta}}^3 V(0)$. Therefore we look for the coefficients of the data

$$(41) \quad \left\{ D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0); \quad |\vec{\alpha}| = j-1 \right\}.$$

in the expansion of $a_j(t)$.

PROPOSITION 2.4. *In the expansion of $a_j(t)$, the coefficient of the data $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0)$, $|\vec{\alpha}| = j-1$, is*

$$(42) \quad \frac{c_2(n)}{(2i)^{j+2}} \frac{t}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(\frac{1}{3\omega_n^2} \left(\frac{2\alpha_n+5}{\alpha_n+1} \right) \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 + \frac{1}{9\omega_n^4} \right).$$

Therefore

$$(43) \quad \begin{aligned} a_j(t) &= \frac{c_1(n)}{(2i)^{j+1}} \sum_{|\vec{\alpha}|=j+1} \frac{t}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} D_{2\vec{\alpha}}^{2j+2} V(0) \\ &+ \frac{c_2(n)}{(2i)^{j+2}} \sum_{|\vec{\alpha}|=j-1} \frac{t}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(\frac{1}{3\omega_n^2} \left(\frac{2\alpha_n+5}{\alpha_n+1} \right) \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 + \frac{1}{9\omega_n^4} \right) D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0) \\ &+ \{a \text{ polynomial of Taylor coefficients of order } \leq 2j\}. \end{aligned}$$

Before we prove Proposition 2.4, let us show how to use this proposition to prove Theorem 1.2.

PROOF OF THEOREM 1.2. First of all, we prove that for all $\vec{\alpha}$, the functions

$$\left(\cot \frac{\vec{\omega}}{2} t \right)^{\vec{\alpha}},$$

are linearly independent over \mathbb{C} . To show this we define

$$\begin{cases} \vec{\cot} : (0, \pi)^n \longrightarrow \mathbb{R}^n, \\ \vec{\cot}(x_1, \dots, x_n) = (\cot(x_1), \dots, \cot(x_n)). \end{cases}$$

Because ω_k are linearly independent over \mathbb{Q} , the set $\{(\frac{\omega_1}{2}t, \dots, \frac{\omega_n}{2}t) + \pi\mathbb{Z}^n; t \in \mathbb{R}\} \cap (0, \pi)^n$ is dense in $(0, \pi)^n$. Since $\vec{\cot}$ is a homeomorphism and is π -periodic, we conclude that the set $\{(\cot(\frac{\omega_1}{2}t), \dots, \cot(\frac{\omega_n}{2}t)); t \in \mathbb{R}\}$ is dense in \mathbb{R}^n . Now assume

$$\sum_{\vec{\alpha}} c_{\vec{\alpha}} \left(\cot \frac{\vec{\omega}}{2} t \right)^{\vec{\alpha}} = 0.$$

Since $\{(\cot(\frac{\omega_1}{2}t), \dots, \cot(\frac{\omega_n}{2}t)); t \in \mathbb{R}\}$ is dense in \mathbb{R}^n , we get

$$\sum_{\vec{\alpha}} c_{\vec{\alpha}} \vec{X}^{\vec{\alpha}} = 0,$$

for every $\vec{X} = (X_1, \dots, X_n) \in \mathbb{R}^n$. But the monomials $\vec{X}^{\vec{\alpha}}$ are linearly independent over \mathbb{C} . So $c_{\vec{\alpha}} = 0$.

Next we argue inductively to recover the Taylor coefficients of V from the wave invariants. Since

$$a_0(t) = \prod_{k=1}^n \frac{1}{2i \sin \frac{\omega_k t}{2}},$$

we can recover $\prod_{k=1}^n \sin \frac{\omega_k t}{2}$, and therefore we can recover $\{\omega_k\}$ up to a permutation. This can be seen by Taylor expanding $\prod_{k=1}^n \sin \frac{\omega_k t}{2}$. We fix this permutation and we move on to recover the third order Taylor coefficient $D_{3\vec{e}_n}^3 V(0)$. This term appears first in $a_1(t)$. By Proposition 2.4, we have

$$\begin{aligned} a_1(t) &= c_1(n) \frac{t}{(2i)^2} \sum_{|\vec{\alpha}|=2} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} D_{2\vec{\alpha}}^4 V(0) \\ &+ c_2(n) \frac{t}{(2i)^3} \left(\frac{5}{3\omega_n^2} \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 + \frac{1}{9\omega_n^4} \right) \left(D_{3\vec{e}_n}^3 V(0) \right)^2 \\ &+ \{a \text{ rational function of } \omega_k\}. \end{aligned}$$

Now since the functions $\{(\cot \frac{\vec{\omega}t}{2})^{\vec{\alpha}}\}_{|\vec{\alpha}|=2}$ and $\left(\frac{5}{3\omega_n^2}(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2})^2 + \frac{1}{9\omega_n^4}\right)$ are linearly independent over \mathbb{C} , we can therefore recover the data $\{D_{2\vec{\alpha}}^4 V(0)\}_{|\vec{\alpha}|=2}$ and $\{D_{3\vec{e}_n}^3 V(0)^2\}$ from $a_1(t)$. So we have determined the third order term $D_{3\vec{e}_n}^3 V(0)$ up to a minus sign from the first invariant $a_1(t)$. This choice of minus sign corresponds to a reflection. We fix this reflection and we move on to determine the higher order Taylor coefficients inductively.

Next we assume $D_{3\vec{e}_n}^3 V(0) \neq 0$ and that we know all the Taylor coefficients $D_{\vec{\beta}}^m V(0)$ with $m \leq 2j$. We wish to determine the data $\{D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0)\}_{|\vec{\alpha}|=j-1}$ and $\{D_{2\vec{\alpha}}^{2j+2} V(0)\}_{|\vec{\alpha}|=j+1}$, from the wave invariant $a_j(t)$. At this point we use Proposition 2.4, and to finish the proof of Theorem 1.1 we have to show that the set of functions

$$\left\{(\cot \frac{\vec{\omega}t}{2})^{\vec{\alpha}}; \quad |\vec{\alpha}| = j+1\right\} \cup \left\{(\cot \frac{\vec{\omega}t}{2})^{\vec{\alpha}} \left(\frac{1}{3\omega_n^2}(\frac{2\alpha_n+5}{\alpha_n+1})(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2})^2 + \frac{1}{9\omega_n^4}\right); \quad |\vec{\alpha}| = j-1\right\},$$

are linearly independent over \mathbb{C} . But this is clear from our discussion at the beginning of proof.

PROOF OF PROPOSITION 2.4. As we mentioned at the beginning of Section 2.7, the data $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0)$, $|\vec{\alpha}| = j-1$, appears first in $a_j(t)$ and it is a part of the term $\int_0^t \int_0^{s_1} P_{j+2} b_2(0)$. So let us calculate those terms in the expansion of $P_{j+2} b_2(0)$ which contain $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0)$. By (20), since here $l = 2$, we have

$$(44) \quad \begin{aligned} b_2(s_1, s_2, x, x, z_1, z_2, \xi_1, \xi_2) &= W_1 W_2, \quad \text{where,} \\ W_1 &= W\left(\frac{\cos \omega_k s_1}{2}(z_1^k + z_2^k) - \frac{\sin \omega_k s_1}{\omega_k} \xi_1^k + \left(\frac{\sin \omega_k (t-s_1) + \sin \omega_k s_1}{\sin \omega_k t}\right) x^k\right), \\ W_2 &= W\left(\frac{\cos \omega_k s_2}{2} z_2^k - \frac{\sin \omega_k s_2}{\omega_k} \xi_2^k + \left(\frac{\sin \omega_k (t-s_2) + \sin \omega_k s_2}{\sin \omega_k t}\right) x^k\right). \end{aligned}$$

Also from (34) we have

$$(45) \quad H_2^{-1} = \begin{pmatrix} D(\frac{-1}{2\vec{\omega}} \cot(\frac{\vec{\omega}t}{2})) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & -I \\ 0 & 0 & 0 & 0 & -I \\ 0 & -I & 0 & -D(\vec{\omega} \cot(\vec{\omega}t)) & -D(\vec{\omega} \cot(\vec{\omega}t)) \\ 0 & -I & -I & -D(\vec{\omega} \cot(\vec{\omega}t)) & -D(\vec{\omega} \cot(\vec{\omega}t)) \end{pmatrix}_{5n \times 5n}.$$

By (37) and (44), $P_{j+2} b_2(0) = \frac{i^{-j}}{2^j j!} S_2$, where S_2 is the following sum

$$(46) \quad S_2 = \sum_{r_1, \dots, r_{2j+4} \in \mathcal{A}_2} h_2^{r_1 r_2} \dots h_2^{r_{2j+3} r_{2j+4}} (W_1 W_2)_{r_1 \dots r_{2j+4}}(0),$$

where $\mathcal{A}_2 = \{x^k, z_1^k, z_2^k, \xi_1^k, \xi_2^k\}_{k=1}^n$ and for every $r, r' \in \mathcal{A}_2$, $h_2^{r r'}$ is the (r, r') -entry of the matrix H_2^{-1} in (45). We would like to separate out those terms in S_2 which include $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0)$. To do this, from the total number $2j+4$ derivatives that we want to apply to $W_1 W_2$, we have to put 3 of them on W_1 (or W_2) and put $2j+1$ of them on W_2 (or W_1 respectively). These combinations fit into one of the following two different forms

$$(47) \quad S_2^1 = \sum_{r_1, \dots, r_{2j+4} \in \mathcal{A}_2} h_2^{r_1 r_2} \dots h_2^{r_{2j+3} r_{2j+4}} (W_1)_{r_1 r_2 r_3} (W_2)_{r_4, r_5 \dots r_{2j+4}}(0).$$

There are $2(j+1)(j+2)$ terms of this form in the expansion of S_2 .

$$(48) \quad S_2^2 = \sum_{r_1, \dots, r_{2j+4} \in \mathcal{A}_2} h_2^{r_1 r_2} \dots h_2^{r_{2j+3} r_{2j+4}} (W_1)_{r_1 r_3 r_5} (W_2)_{r_2, r_4, r_6, r_7 \dots r_{2j+4}} (0).$$

There are $2^3 \binom{j+2}{3}$ terms of this form in the expansion of S_2 .

Now, we calculate the sums S_2^1 and S_2^2 .

2.8. Calculation of S_2^1 . We rewrite S_2^1 as

$$S_2^1 = \sum_{r_1, \dots, r_4} h_2^{r_1 r_2} h_2^{r_3 r_4} \left(\sum_{r_5, \dots, r_{2j+4}} h_2^{r_5 r_6} \dots h_2^{r_{2j+3} r_{2j+4}} (W_2)_{r_5 \dots r_{2j+4}} \right)_{r_4} (W_1)_{r_1 r_2 r_3} (0).$$

Then from the definition of W_2 in (44) and also from (45) it is clear that we can apply (40) to the sum in the big parenthesis above. Hence we get

$$(49) \quad S_2^1 = (j+1)! \sum_{|\vec{\alpha}|=j} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(\sum_{r_1, \dots, r_4} h_2^{r_1 r_2} h_2^{r_3 r_4} (D_{2\vec{\alpha}}^{2j} W_2)_{r_4} (W_1)_{r_1 r_2 r_3} \right) (0).$$

This reduces the calculation of S_2^1 to calculating the small sum

$$A_2^1 = \sum_{r_1, \dots, r_4} h_2^{r_1 r_2} h_2^{r_3 r_4} (\tilde{W}_2)_{r_4} (W_1)_{r_1 r_2 r_3} (0), \quad (\tilde{W}_2 = D_{2\vec{\alpha}}^{2j} W_2).$$

Computation of the sum A_2^1 is straight forward and we omit writing the details of this computation. Using Maple, we obtain

$$\int_0^t \int_0^{s_1} A_2^1 ds_2 dt = -\frac{t}{2\omega_n^2} \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right) (D_{\vec{e}_n}^1 \tilde{W}_2 D_{3\vec{e}_n}^3 W_1) (0).$$

If we plug this into (49), after a change of variable $\alpha_n \rightarrow \alpha_n + 1$ in indices, we get

$$(50) \quad \int_0^t \int_0^{s_1} S_2^1 ds_2 dt = \frac{(j+1)!}{\alpha_n + 1} \sum_{|\vec{\alpha}|=j-1} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(-\frac{t}{2\omega_n^2} \right) \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0).$$

2.9. Calculation of S_2^2 . We rewrite S_2^2 as

$$S_2^2 = \sum_{r_1, \dots, r_6} h_2^{r_1 r_2} h_2^{r_3 r_4} h_2^{r_5 r_6} \left(\sum_{r_7, \dots, r_{2j+4}} h_2^{r_7 r_8} \dots h_2^{r_{2j+3} r_{2j+4}} (W_2)_{r_7 \dots r_{2j+4}} \right)_{r_2, r_4, r_6} (W_1)_{r_1 r_3 r_5} (0).$$

Again from (45) it is clear that we can apply (40) to the sum in the big parenthesis above. So

$$(51) \quad S_2^2 = (j+1)! \sum_{|\vec{\alpha}|=j-1} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(\sum_{r_1, \dots, r_6} h_2^{r_1 r_2} h_2^{r_3 r_4} (D_{2\vec{\alpha}}^{2j-2} W_2)_{r_2, r_4, r_6} (W_1)_{r_1 r_3 r_5} \right) (0).$$

So we need to compute

$$A_2^2 = \sum_{r_1, \dots, r_6} h_2^{r_1 r_2} h_2^{r_3 r_4} h_2^{r_5 r_6} (\tilde{W}_2)_{r_2, r_4, r_6} (W_1)_{r_1 r_3 r_5} (0), \quad (\tilde{W}_2 = D_{2\vec{\alpha}}^{2j-2} W_2).$$

Using Maple

$$\int_0^t \int_0^{s_1} A_2^2 ds_2 dt = \left(-\frac{t}{2\omega_n^2} \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 - \frac{t}{12\omega_n^4} \right) (D_{3\vec{e}_n}^3 \tilde{W}_2 D_{3\vec{e}_n}^3 W_1) (0).$$

If we plug this into (51) we get

$$(52) \quad \int_0^t \int_0^{s_1} S_2^2 ds_2 dt = (j+1)! \sum_{|\vec{\alpha}|=j-1} \frac{1}{\vec{\alpha}!} \left(\frac{-1}{2\vec{\omega}} \cot \frac{\vec{\omega}t}{2} \right)^{\vec{\alpha}} \left(-\frac{t}{2\omega_n^2} \left(\frac{-1}{2\omega_n} \cot \frac{\omega_n t}{2} \right)^2 - \frac{t}{12\omega_n^4} \right) D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0).$$

We note that the part of the expansion of $\int_0^t \int_0^{s_1} P_{j+2} b_2(0)$ which contains the data $D_{2\vec{\alpha}+3\vec{e}_n}^{2j+1} V(0) D_{3\vec{e}_n}^3 V(0)$, equals

$$\frac{i^{-j}}{2^j \cdot j!} \left(2(j+2)(j+1) \int_0^t \int_0^{s_1} S_2^1 + 2^3 \binom{j+2}{3} \int_0^t \int_0^{s_1} S_2^2 \right).$$

Finally, by applying equations (50) and (52) to this we obtain (43).

3. APPENDIX A

In this appendix we prove Lemma 2.1.

PROOF. First of all we would like to change the function Θ slightly by rescaling it. We choose $0 < \tau < 2\varepsilon$ so that $\hbar^{1-\tau} = o(\hbar^{1-2\varepsilon})$. Then we define

$$(53) \quad \Theta_{\hbar}(x) := \Theta\left(\frac{x}{\hbar^{1-\tau}}\right).$$

Thus $\Theta_{\hbar} \in C_0^\infty([0, \infty))$ is supported in the interval $I_{\hbar} = [0, \hbar^{1-\tau}\delta]$. In Appendix B, using min-max principle we show that

$$Tr(\Theta(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}) = Tr(\Theta_{\hbar}(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}) + O(\hbar^\infty) = Tr(e^{\frac{-it}{\hbar}\hat{H}}) + O(\hbar^\infty).$$

Hence to prove the lemma it is enough to show

$$Tr(\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) = Tr(\Theta_{\hbar}(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}) + O(\hbar^\infty).$$

To prove this identity we use the WKB construction of the kernel of the operators $\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}$ and $\Theta_{\hbar}(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}$ and make a compression between them.

3.1. WKB construction for $\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}$. In [DSj], chapter 10, a WKB construction is made for $\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}$ for symbols P in the symbol class $S_0^0(1)$ which are independent of \hbar or of the form $P(x, \xi, \hbar) \sim P_0(x, \xi) + \hbar P_1(x, \xi) + \dots$, where $P_j \in S_0^0$ are independent of \hbar (but not for symbols $H = H(x, \xi, \hbar) \in S_{00}^0$).

It is shown that we can approximate $\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}}$ for small time t , say $t \in (-t_0, t_0)$, by a fourier integral operator of the form

$$U_P(t)u(x) = (2\pi\hbar)^{-n} \int \int e^{i(\varphi_P(t, x, \eta) - y \cdot \eta)/\hbar} b_P(t, x, y, \eta, \hbar) u(y) dy d\eta,$$

where $b_P \in C^\infty((-t_0, t_0); S(1))$ have uniformly compact support in (x, y, η) , and φ_P is real, smooth and is defined near the support of b_P . The functions φ_P and b_P are found in such a way that for all $t \in (-t_0, t_0)$

$$\|\Theta(\hat{P})e^{\frac{-it}{\hbar}\hat{P}} - U_P(t)\|_{tr} = O(\hbar^\infty).$$

Let us briefly review this construction, made in [DSj]. First of all, in Chapter 8, Theorem 8.7, it is proved that for every symbol $P \in S_0^0(1)$, we have $\Theta(\hat{P}) = Op^w(a_P(x, \xi, \hbar))$ for some $a_P(x, \xi, \hbar) \in S_0^0(1)$, where here \hat{P} and $Op^w(a_P(x, \xi, \hbar))$ are respectively the Weyl quantization of P and $a_P(x, \xi, \hbar)$. It is also shown that $a_P \sim a_{P,0}(x, \xi) + \hbar a_{P,1}(x, \xi) + \dots$ for some $a_{P,j}(x, \xi) \in S_0^0(1)$. The idea of proof is as follows. In Theorem 8.1 of [DSj] it is shown that if $\Theta \in C_0^\infty(\mathbb{R})$, and if $\tilde{\Theta} \in C_0^1(\mathbb{C})$ is an almost analytic extension of Θ (i.e. $\bar{\partial}\tilde{\Theta}(z) = O(|\Im z|^\infty)$), then

$$(54) \quad \Theta(\hat{P}) = \frac{-1}{\pi} \int_{\mathbb{C}} \frac{\bar{\partial}\tilde{\Theta}(z)}{z - \hat{P}} L(dz).$$

Then it is verified that for some symbol $r(x, \xi, z; \hbar)$, we have $(z - \hat{P})^{-1} = Op^w(r(x, \xi, z; \hbar))$. By symbolic calculus, one can find a formal asymptotic expansion of the form

$$r(x, \xi, z; \hbar) \sim \frac{1}{z - P} + \hbar \frac{q_1(x, \xi, z)}{(z - P)^3} + \hbar^2 \frac{q_2(x, \xi, z)}{(z - P)^5} + \dots,$$

by formally solving $Op^w(r(x, \xi, z; \hbar))\sharp_\hbar(z - \hat{P}) = (z - \hat{P})\sharp_\hbar Op^w(r(x, \xi, z; \hbar)) = 1$. We can see that $q_j(x, \xi, z)$ are polynomials in z with smooth coefficients. Finally it is shown that $\Theta(\hat{P}) = Op^w(a_P(x, \xi, \hbar))$, where $a_P \in S_0^0$ is given by

$$a_P(x, \xi, \hbar) = \frac{-1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\Theta}(z) r(x, \xi, z; \hbar) L(dz).$$

By the above asymptotic expansion for $r(x, \xi, z; \hbar)$ one obtains an asymptotic $a_P \sim a_{P,0} + \hbar a_{P,1} + \dots$, where

$$(55) \quad a_{P,j} = \frac{-1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\Theta}(z) \frac{q_j(x, \xi, z)}{(z - P)^{2j+1}} L(dz) = \frac{1}{(2j)!} \partial_t^{2j} (q_j(x, \xi, t) \Theta(t))|_{t=P(x, \eta)}.$$

Then, again in Chapter 10 of [DSj], it is shown that $\varphi_P(t, x, \eta)$ and $b_P(t, x, y, \eta, \hbar)$ satisfy

$$(56) \quad \partial_t \varphi_P(t, x, \eta) + P(x, \partial_x \varphi_P(t, x, \eta)) = 0, \quad \varphi_P|_{t=0} = x \cdot \eta,$$

$$b_P \sim b_{P,0} + \hbar b_{P,1} + \dots, \quad b_{P,j} = b_{P,j}(t, x, y, \eta) \in C^\infty((-t_0, t_0); S_0^0(1)),$$

where

$$(57) \quad \begin{cases} \partial_t b_{P,j} + \langle \partial_x \varphi_P, \partial_x b_{P,j} \rangle + \frac{1}{2} \Delta_x \varphi_P \cdot b_{P,j} = -\frac{1}{2} \Delta_x b_{P,j-1}, & j \geq 0, & (b_{P,-1} = 0), \\ b_{P,j}|_{t=0} = \psi(x, \eta) a_{P,j}(\frac{x+y}{2}, \eta) \psi(y, \eta). \end{cases}$$

In (57), $a_{P,j}$ is given by (55) and $\psi(x, \eta)$ is any C_0^∞ function which equals 1 in a neighborhood of $\overline{P^{-1}(I)}$ where $I = [0, \delta]$ is, as before, the range of our low-lying eigenvalues and where Θ is supported.

There exists a similar construction for $\Theta_\hbar(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}$, except here $H \in S_{\delta_0}^0$.

3.2. WKB construction for $\Theta_\hbar(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}$. Since in (13), $H = H(x, \xi, \hbar) \in S_{\delta_0}^0$, with $\delta_0 = \frac{1}{2} - \varepsilon$, we can not simply use the construction in [DSj] mentioned above. Here in two lemmas we show that the same construction works for the operator $\Theta_\hbar(\hat{H})e^{\frac{-it}{\hbar}\hat{H}}$. We will closely follow the proofs in [DSj].

LEMMA 3.1.

- 1) Let Θ_\hbar be given by (53) and $H \in S_{\delta_0}^0$ by (13). Then for some $a_H \in S_{\delta_0}^0$ we have $\Theta(\hat{H}) = Op^w(a_H(x, \xi, \hbar))$. Moreover $a_H(x, \xi, \hbar) \sim a_{H,0}(x, \xi, \hbar) + \hbar a_{H,1}(x, \xi, \hbar) + \dots$, where $a_{H,j}(x, \xi, \hbar) \in S_{\delta_0}^0$ is given by

$$(58) \quad a_{H,j} = \frac{-1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{\Theta}(z) \frac{q_{H,j}(x, \xi, z, \hbar)}{(z - H)^{2j+1}} L(dz) = \frac{1}{(2j)!} \partial_t^{2j} (q_{H,j}(x, \xi, t, \hbar) \Theta_\hbar(t))|_{t=H(x, \xi, \hbar)}.$$

- 2) Choose c such that $0 < c < \min\{1, \omega_k^2\}_{k=1}^n \leq \max\{1, \omega_k^2\}_{k=1}^n < \frac{1}{c}$. Let $\psi_\hbar(x, \eta)$ be a function in $C_0^\infty(\mathbb{R}^{2n}) \cap S_{\delta_0}^0(\mathbb{R}^{2n})$ which is supported in the ball $\{x^2 + \eta^2 < 4c^{-1}\hbar^{1-\tau}\delta\}$ and equals 1 in a neighborhood of $\overline{H^{-1}(I)}$, where $I_\hbar = [0, \hbar^{1-\tau}\delta]$ (I_\hbar is where Θ_\hbar is supported). Then

$$(59) \quad \Theta_\hbar(\hat{H})u(x) = (2\pi\hbar)^{-n} \int \int e^{i(x-y)\cdot\eta/\hbar} \psi_\hbar(x, \eta) a_H(\frac{x+y}{2}, \eta, \hbar) \psi_\hbar(y, \eta) u(y) dy d\eta + K(\hbar)u(x),$$

where $\|K(\hbar)\|_{tr} = O(\hbar^\infty)$.

PROOF OF LEMMA 3.1: Since $H \in S_{\delta_0}^0$ and $\delta_0 = \frac{1}{2} - \varepsilon < \frac{1}{2}$, the symbolic calculus mentioned in the last section can be followed similarly to prove Lemma 3.1.1. It is also easy to check that in (58), $a_{H,j} \in S_{\delta_0}^0$. The second part of the Lemma is stated in [DSj], equation 10.1, for the case $P \in S_0^0$. The same argument works for $H \in S_{\delta_0}^0$, precisely because the factor \hbar^N on the right hand side of the inequality in Proposition 9.5 of [DSj] changes to $\hbar^{N-\delta_0\alpha}$. Thus the discussion on pages 115 – 116 still follows.

LEMMA 3.2. *For every t in some small interval $(-t_0, t_0)$, there exist functions $\varphi_H(t, x, \eta, \hbar)$ and $b_H(t, x, y, \eta, \hbar)$ such that the operator $U_H(t)$ defined by*

$$(60) \quad U_H(t)u(x) = (2\pi\hbar)^{-n} \int \int e^{i(\varphi_H(t, x, \eta, \hbar) - y \cdot \eta)/\hbar} b_H(t, x, y, \eta, \hbar) u(y) dy d\eta,$$

satisfies

$$\|\Theta_\hbar(\hat{H})e^{-\frac{it}{\hbar}\hat{H}} - U_H(t)\|_{tr} = O(\hbar^\infty).$$

Moreover, we can choose φ_H and b_H such that

1) φ_H satisfies the eikonal equation

$$(61) \quad \partial_t \varphi_H(t, x, \eta, \hbar) + H(x, \partial_x \varphi_H(t, x, \eta, \hbar)) = 0, \quad \varphi_H|_{t=0} = x \cdot \eta.$$

This equation can be solved in $(-t_0, t_0) \times \{x^2 + \eta^2 < C\hbar^{1-\tau}\delta\}$ where C is an arbitrary constant. In fact φ_H is independent of \hbar in this domain. (Only the domain of φ_H depends on \hbar . See (64).)

2) For all $t \in (-t_0, t_0)$, we have $b_H(t, x, y, \eta, \hbar) \in S_{\delta_0}^0$ with $\text{supp } b_H \subset \{x^2 + \eta^2, y^2 + \eta^2 < C_1\hbar^{1-\tau}\delta\}$ for some constant C_1 . Also b_H has an asymptotic expansion of the form

$$(62) \quad b_H \sim b_{H,0} + \hbar b_{H,1} + \dots, \quad b_{H,j} = b_{H,j}(t, x, y, \eta, \hbar) \in C^\infty((-t_0, t_0); S_{\delta_0}^0(1)),$$

and the functions $b_{H,j}$ satisfy the transport equations

$$(63) \quad \begin{cases} \partial_t b_{H,j} + \langle \partial_x \varphi_H, \partial_x b_{H,j} \rangle + \frac{1}{2} \Delta_x \varphi_H \cdot b_{H,j} = -\frac{1}{2} \Delta_x b_{H,j-1}, & j \geq 0, \quad (b_{H,-1} = 0), \\ b_{H,j}|_{t=0} = \psi_\hbar(x, \eta) a_{H,j}(\frac{x+y}{2}, \eta, \hbar) \psi_\hbar(y, \eta), \end{cases}$$

where in (63) we let $\psi_\hbar(x, \eta)$ be a function in $C_0^\infty(\mathbb{R}^{2n}) \cap S_{\delta_0}^0(\mathbb{R}^{2n})$ which is supported in the ball $\{x^2 + \eta^2 < 4c\hbar^{1-\tau}\delta\}$ and equals 1 in a neighborhood of $\overline{H^{-1}(I_\hbar)}$, where $I_\hbar = [0, \hbar^{1-\tau}\delta]$. Here c is defined in Lemma 3.1.2. Also in (63), the functions $a_{H,j}$ are defined by (58).

4)

$$(64) \quad \varphi_H(t, x, \eta, \hbar) = \varphi_P(t, x, \eta) \quad \text{on} \quad \{x^2 + \eta^2, y^2 + \eta^2 < C_1\hbar^{1-\tau}\delta\} \supset \text{supp}(b_H(x, y, \eta, \hbar)).$$

5)

$$(65) \quad b_{H,j}(t, x, y, \eta, \hbar) = b_{P,j}(t, x, y, \eta) \quad \text{on} \quad \{x^2 + \eta^2, y^2 + \eta^2 < c\hbar^{1-\tau}\delta\}.$$

PROOF OF LEMMA 3.2: First of all we assume $U_H(t)$ is given by (60) and we try to solve the equation

$$\begin{cases} \|(\frac{\hbar}{i}\partial_t + \hat{H})U_H(t)\|_{tr} = O(\hbar^\infty), \\ U_H(0) = \Theta_\hbar(\hat{H}). \end{cases}$$

for φ_H and b_H , for small time t .

Using (59), this leads us to

$$\begin{cases} e^{-i\varphi_H/\hbar}(\frac{\hbar}{i}\partial_t + \hat{H})(e^{i\varphi_H/\hbar}b_H) \in C^\infty((-t_0, t_0); S_{\delta_0}^{-\infty}(1)), \\ b|_{t=0} = \psi_\hbar(x, \eta)a_H(\frac{x+y}{2}, \eta, \hbar)\psi_\hbar(y, \eta). \end{cases}$$

We choose the phase function $\varphi_H = \varphi_H(t, x, \eta, \hbar)$ to satisfy the eikonal equation (61). This equation can be solved in a neighborhood of the support of b_H , for small time $t \in (-t_0, t_0)$ with t_0 independent of \hbar . Let us explain how to solve this equation. We let $(x(t, z, \eta; \hbar), \xi(t, z, \eta; \hbar))$ be the solution to the Hamilton equation

$$(66) \quad \begin{cases} \partial_t x = \partial_\xi H(x, \xi, \hbar) = \xi, & x(0, z, \eta; \hbar) = z \\ \partial_t \xi = -\partial_x H(x, \xi, \hbar) = -\partial_x V_\hbar(x), & \xi(0, z, \eta; \hbar) = \eta \end{cases}.$$

We can show that (see section 4 of [Ch]) there exists t_0 independent of \hbar such that for all $|t| \leq t_0$ we have

$$(67) \quad \begin{cases} |\partial_z x(t, z, \eta; \hbar) - I| \leq \frac{1}{2}, & |\partial_\eta x(t, z, \eta; \hbar)| \leq \frac{1}{2} \\ |\partial_z \xi(t, z, \eta; \hbar)| \leq \frac{1}{2}, & |\partial_\eta \xi(t, z, \eta; \hbar) - I| \leq \frac{1}{2} \end{cases}.$$

We can choose t_0 independent of \hbar , precisely because in equation 4.4 of [Ch] we have a uniform bound in \hbar for $\text{Hess}(V_\hbar(x))$. Now, we define

$$\lambda : (z, \eta) \mapsto (x(t, z, \eta; \hbar), \eta).$$

It is easy to see that $\lambda(0, 0) = (0, 0)$. This is because if $(z, \eta) = (0, 0)$ then $H(x, \xi) = H(z, \eta) = 0$. By (13) and (2), and $W(x) = O(|x|^3)$, we can see that $H(x, \xi) = 0$ implies $(x(t, 0, 0; \hbar), \xi(t, 0, 0; \hbar)) = (0, 0)$. On the other hand from (67) we have $\frac{1}{2} < |\partial_z x(t, z, \eta; \hbar)| < \frac{3}{2}$. Therefore λ is invertible in a neighborhood of origin. We define the inverse function by

$$\lambda^{-1}(x, \eta) = (z(t, x, \eta; \hbar), \eta),$$

which is defined in a neighborhood of $(x, \eta) = (0, 0)$. Then we have

$$(68) \quad \varphi_H(t, x, \eta, \hbar) = z(t, x, \eta; \hbar) \cdot \eta + \int_0^t \frac{1}{2} |\xi(s, z(t, x, \eta; \hbar), \eta; \hbar)|^2 - V_\hbar(x(s, z(t, x, \eta; \hbar), \eta; \hbar)) ds,$$

A similar formula holds for φ_P except in (66) H should be replaced by P and in (68) V_\hbar by V . It is known that the eikonal equation for φ_P can be solved near $\text{supp} b_P$, for small time $t \in (-t_0, t_0)$ (Of course t_0 is independent of \hbar). Now, we want to show that

$$(69) \quad \varphi_H(t, x, \eta, \hbar) = \varphi_P(t, x, \eta) \quad \text{in} \quad (-t_0, t_0) \times \{x^2 + \eta^2 < C\hbar^{1-\tau}\delta\}.$$

Let (x, η) be in $\{x^2 + \eta^2 < C\hbar^{1-\tau}\delta\}$. First, we show that $|z(t, x, \eta; \hbar)| < 8C^{\frac{1}{2}}\hbar^{\frac{1-\tau}{2}}\delta^{\frac{1}{2}}$. Because $z(t, 0, 0; \hbar) = 0$, by Fundamental Theorem of Calculus we have

$$|z(t, x, \eta; \hbar)| \leq (|x| + |\eta|) \sup\{(|\partial_x| + |\partial_\eta|)(z(t, x, \eta; \hbar))\}.$$

From $x(t, z(t, x, \eta; \hbar), \eta; \hbar) = x$, we get

$$\partial_\eta z = -(\partial_z x)^{-1} \partial_\eta x.$$

Thus by (67), $|\partial_x z| + |\partial_\eta z| \leq 4$. Hence $|z(t, x, \eta; \hbar)| < 4(|x| + |\eta|) < 8C^{\frac{1}{2}}\hbar^{\frac{1-\tau}{2}}\delta^{\frac{1}{2}}$. This implies that for all $|t| \leq t_0$, $(x(s, z(t, x, \eta; \hbar), \eta; \hbar), \xi(s, z(t, x, \eta; \hbar), \eta; \hbar))$ will stay in a ball of radius $O(\hbar^{1-\tau})$ centered at the origin. On the other hand, by definition (13), P and H agree in the ball $\{x^2 + \eta^2 < \frac{1}{4}\hbar^{1-2\varepsilon}\}$ and $\tau < 2\varepsilon$. So for all $t, s \in (-t_0, t_0)$ and $(x, \eta) \in \{x^2 + \eta^2 < C\hbar^{1-\tau}\delta\}$ we have

$$(70) \quad \begin{aligned} z_P(t, x, \eta) &= z(t, x, \eta; \hbar), \\ x_P(s, z_P(t, x, \eta), \eta) &= x(s, z(t, x, \eta; \hbar), \eta; \hbar), \\ \xi_P(s, z_P(t, x, \eta), \eta) &= \xi(s, z(t, x, \eta; \hbar), \eta; \hbar), \end{aligned}$$

where $z_P(t, x, \eta)$, $x_P(s, z_P(t, x, \eta), \eta)$ and $\xi_P(s, z_P(t, x, \eta), \eta)$ are corresponded to the Hamilton flow of P . Hence by (68) and a similar formula for φ_P , we have (69). This also shows that we can solve (61) in $(-t_0, t_0) \times \{x^2 + \eta^2 < C\hbar^{1-\tau}\delta\}$.

To find b_H we assume it is of the form (62) and we search for functions $b_{H,j}$ such that $e^{-i\varphi_H/\hbar}(\frac{\hbar}{i}\partial_t + \hat{H})(e^{i\varphi_H/\hbar}b_H) \sim 0$. After some straightforward calculations and using the eikonal equation for φ_H we obtain the so called transport equations (63). Now let us solve the transport equations inductively (see [Ch]).

In [Ch] it is shown that the solutions to the transport equation (63) are given by

$$(71) \quad \begin{aligned} b_{H,0}(t, x, y, \eta, \hbar) &= J^{-\frac{1}{2}}(t, x, \eta, \hbar) b_{H,0}(0, z(t, x, \eta; \hbar), \eta; \hbar), y, \eta, \hbar) \\ b_{H,j}(t, x, y, \eta, \hbar) &= J^{-\frac{1}{2}}(t, x, \eta, \hbar) \left(b_{H,j}(0, z(t, x, \eta; \hbar), \eta; \hbar), y, \eta, \hbar) \right. \\ &\quad \left. - \frac{1}{2} \int_0^t J^{\frac{1}{2}}(s, x, \eta, \hbar) \Delta b_{H,j-1}(s, x(s, z(t, x, \eta; \hbar), \eta; \hbar), y, \eta, \hbar) ds \right). \end{aligned}$$

where

$$J(t, x, \eta, \hbar) = \det(\partial_x z(t, x, \eta; \hbar))^{-1}.$$

Now, we notice by the assumption on ψ_h , we have $\text{supp}(b_{H,j}(0, x, y, \eta; \hbar)) \subset \{x^2 + \eta^2, x^2 + \eta^2 < 4c^{-1}\hbar^{1-\tau}\delta\}$. So by our previous discussion on $z(t, x, \eta, \hbar)$, we can argue inductively that for all $t \in (-t_0, t_0)$, $\text{supp}(b_{H,j}) \subset \{x^2 + \eta^2, y^2 + \eta^2 < C_1\hbar^{1-\tau}\delta\}$ for some constant C_1 . Since $b_{H,j}|_{t=0} \in S_{\delta_0}^0$, we can also see inductively from (71) that $b_{H,j} \in S_{\delta_0}^0$. Finally, Borel's theorem produces a compactly supported amplitude $b_H \in S_{\delta_0}^0$ from the compactly supported functions $b_{H,j} \in S_{\delta_0}^0$. This finishes the proof of items 1 – 3 in Lemma 3.2.

Now we give proofs for items 4, 5 in Lemma 3.2.

By choosing $C > C_1$, equation (64) is clearly true from (69). Next we prove that equation (65) holds. Using (55) and (58), and because P and H agree in the ball $\{x^2 + \eta^2 < \frac{1}{4}\hbar^{1-2\varepsilon}\}$, we observe that the functions $a_{P,j}(x, \eta)$ and $a_{H,j}(x, \xi, \hbar)$ agree in this ball. Therefore, because $\text{supp}\psi_h(x, \eta) \subset \{x^2 + \eta^2 < 4c^{-1}\hbar^{1-\tau}\delta\}$ and $\psi_h = 1$ in $\{x^2 + \eta^2 < c\hbar^{1-\tau}\delta\}$, by (57) and (63)

$$b_{H,j}(0, x, y, \eta, \hbar) = b_{P,j}(0, x, y, \eta) \quad \text{on} \quad \{(x, y, \eta); x^2 + \eta^2, y^2 + \eta^2 < c\hbar^{1-\tau}\delta\}.$$

This proves (65) only at $t = 0$. But by applying (70) to (71) and a similar formula for b_P , we get (65). This finishes the proof of Lemma 3.2.

To finish the proof of Lemma 2.1, we have to show that for t sufficiently small $\text{Tr}U_H(t) = \text{Tr}U_P(t) + O(\hbar^\infty)$, or equivalently

$$\int \int e^{i(\varphi_H(t, x, \eta, \hbar) - x \cdot \eta)/\hbar} b_H(t, x, x, \eta, \hbar) dx d\eta = \int \int e^{i(\varphi_P(t, x, \eta) - x \cdot \eta)/\hbar} b_P(t, x, x, \eta, \hbar) dx d\eta + O(\hbar^\infty).$$

By (64), the phase function φ_H of the double integral on the left hand side equals φ_P on the support of the amplitude b_H , so φ_H is independent of \hbar in this domain. Now, if $t \in (0, t_0)$ where t_0 is smaller than the smallest non-zero period of the flows of P and H respectively in the energy balls

$$\{(x, \eta) \mid H(x, \eta) \leq \delta\hbar^{1-\tau}\} \subset \{(x, \eta) \mid P(x, \eta) \leq \delta\},$$

then for every such t , $(x, \eta) = (0, 0)$ is the only critical point of the phase functions $\varphi_H(t, x, \eta, \hbar) - x \cdot \eta$ and $\varphi_P(t, x, \eta) - x \cdot \eta$ in these energy balls.

Obviously both integrals in the equation above are convergent because their amplitudes are compactly supported. But the question is whether or not we can apply the stationary phase lemma to these integrals around their unique non-degenerate critical points. By Lemma 3.2 the phase functions φ_H and φ_P are independent of \hbar on the support of their corresponding amplitudes. Hence $\varphi_H, \varphi_P \in S_0^0$ on $\text{supp}b_H$ and $\text{supp}b_P$ respectively. On the other hand $b_H(t, x, x, \eta, \hbar) \in S_{\delta_0}^0$, $\delta_0 < \frac{1}{2}$; and $b_P(t, x, x, \eta, \hbar) \in S_0^0$. These facts can be used to get the required estimates for the remainder term in the stationary phase lemma (for an estimate for the remainder term of the stationary phase lemma, see for example Proposition 5.2 of [DSj]).

Finally, by (64) and (65) it is obvious that the integrals above must have the same stationary phase expansions.

4. APPENDIX B

In this appendix we prove Lemma 2.2. In fact we prove that if Θ_h is given by (53) then in the sense of tempered distributions

$$(72) \quad \text{Tr}(\Theta_h(\hat{H})e^{\frac{-it}{h}\hat{H}}) = \text{Tr}(e^{\frac{-it}{h}\hat{H}}) + O(h^\infty).$$

Proof of Lemma 2.2 follows similarly.

We will use the min-max principle.

Min-max principle. Let H be a self-adjoint operator that is bounded from below, i.e. $H \geq cI$, with purely discrete spectrum $\{E_j\}_{j=0}^\infty$. Then

$$(73) \quad E_j = \sup_{\varphi_1, \dots, \varphi_{n-1}} \inf_{\substack{\psi \in D(H); \|\psi\| = 1 \\ \psi \in \text{span}(\varphi_1, \dots, \varphi_{n-1})^\perp}} (\psi, H\psi).$$

As before we put $\hat{H} = -\frac{1}{2}\hbar^2\Delta + V_h(x) = -\frac{1}{2}\hbar^2\Delta + \frac{1}{2}\sum_{k=1}^n \omega_k^2 x_k^2 + W_h(x)$, and $\hat{H}_0 = -\frac{1}{2}\hbar^2\Delta + \frac{1}{2}\sum_{k=1}^n \omega_k^2 x_k^2$. Then if we let $C = \|W_h(x)\|_{L^\infty(\mathbb{R}^n \times (0, h_0))}$, we have

$$(\psi, \hat{H}_0\psi) - C \leq (\psi, \hat{H}\psi) \leq (\psi, \hat{H}_0\psi) + C,$$

and therefore by applying the min-max principle to the operators \hat{H} and \hat{H}_0 we get

$$(74) \quad E_j^0(\hbar) - C \leq E_j(\hbar) \leq E_j^0(\hbar) + C.$$

Notice we have explicit formulas for the eigenvalues $E_j^0(\hbar)$ of \hat{H}_0 . They are given by the lattice points in the first quadrant of \mathbb{R}^n . More precisely

$$\sigma(\hat{H}_0) = \left\{ E_j^0(\hbar) = \hbar \sum_{k=1}^n \omega_k \left(\gamma_k + \frac{1}{2} \right); \quad \gamma_k \in \mathbb{Z}^{\geq 0} \right\}.$$

Since in the sense of tempered distributions

$$\text{Tr}(\Theta_h(\hat{H})e^{\frac{-it}{h}\hat{H}}) = \text{Tr}(\chi_{[0, \delta\hbar^{1-\tau}]}(\hat{H})e^{\frac{-it}{h}\hat{H}}) + O(h^\infty); \quad (\text{see for example [KD]}),$$

to prove (72), it is clearly enough to show that for every φ in $S(\mathbb{R})$

$$\sum_{\{j; E_j(\hbar) > \delta\hbar^{1-\tau}\}} \hat{\varphi}\left(\frac{E_j(\hbar)}{\hbar}\right) = O(h^\infty).$$

Since $\hat{\varphi}$ is in $S(\mathbb{R})$, for every $p \geq 0$ there exists a constant C_p such that

$$|\hat{\varphi}(x)| \leq C_p |x|^{-p}.$$

Hence by (74)

$$\varphi\left(\frac{E_j(\hbar)}{\hbar}\right) \leq C_p \left| \frac{E_j(\hbar)}{\hbar} \right|^{-p} \leq C_p \left| \frac{E_j^0(\hbar) - C}{\hbar} \right|^{-p}.$$

Again using (74) and because $C = \|W_h(x)\|_{L^\infty(\mathbb{R}^n \times (0, h_0))} < A\hbar^{\frac{3}{2}-3\varepsilon} < \frac{\delta}{4}\hbar^{1-\tau}$ we get

$$\varphi\left(\frac{E_j(\hbar)}{\hbar}\right) \leq C_p \left(\frac{\delta\hbar^{1-\tau} - C}{\delta\hbar^{1-\tau} - 2C} \right)^p \left| \frac{E_j^0(\hbar)}{\hbar} \right|^{-p} < 2C_p \left| \frac{E_j^0(\hbar)}{\hbar} \right|^{-p}, \quad \text{for } E_j(\hbar) > \delta\hbar^{1-\tau}.$$

Now let m be an arbitrary positive integer. So in order to prove the lemma it is enough to find a uniform bound for

$$A(\hbar) := \hbar^{-m} \sum_{\{\tilde{\gamma}; \sum \omega_k(\gamma_k + \frac{1}{2}) > \frac{\delta \hbar^{1-\tau} - C}{n\hbar}\}} \left| \sum_{k=1}^n \omega_k(\gamma_k + \frac{1}{2}) \right|^{-p}.$$

By applying the geometric-arithmetic mean value inequality we get

$$\begin{aligned} A(\hbar) &\leq n^{-p} \hbar^{-m} \sum_{\{\tilde{\gamma}; \sum \omega_k(\gamma_k + \frac{1}{2}) > \frac{\delta \hbar^{1-\tau} - C}{n\hbar}\}} \left| \prod_{k=1}^n \omega_k(\gamma_k + \frac{1}{2}) \right|^{-p} \\ &\leq n^{-p} \sum_{k=1}^n \left\{ \left(\hbar^{-m} \sum_{\{\gamma_k \in \mathbb{Z}^{\geq 0}; \omega_k(\gamma_k + \frac{1}{2}) > \frac{\delta \hbar^{1-\tau} - C}{n\hbar}\}} |\omega_k(\gamma_k + \frac{1}{2})|^{-p} \right) \prod_{k' \neq k} \left(\sum_{\gamma_{k'}} |\omega_{k'}(\gamma_{k'} + \frac{1}{2})|^{-p} \right) \right\}. \end{aligned}$$

We claim for p large enough there is a uniform bound for the sum on the right hand side of the above inequality. It is clear that if $p \geq 2$ then the series $\sum_{\gamma_{k'}} |\omega_{k'}(\gamma_{k'} + \frac{1}{2})|^{-p}$ is convergent. Also if for some γ_k we have $\omega_k(\gamma_k + \frac{1}{2}) > \frac{\delta \hbar^{1-\tau} - C}{n\hbar}$, then because $C = O(\hbar^{\frac{3}{2}-3\varepsilon})$, for \hbar small enough we have $(\omega_k(\gamma_k + \frac{1}{2}))^{1/\tau} > (\frac{\delta}{2n})^{1/\tau} \frac{1}{\hbar}$. Thus

$$\sum_{\{\gamma_k \in \mathbb{Z}^{\geq 0}; \omega_k(\gamma_k + \frac{1}{2}) > \frac{\delta \hbar^{1-\tau} - C}{n\hbar}\}} \hbar^{-m} |\omega_k(\gamma_k + \frac{1}{2})|^{-p} \leq \left(\frac{2n}{\delta}\right)^{m/\tau} \sum_{\gamma_k} |\omega_k(\gamma_k + \frac{1}{2})|^{\frac{m}{\tau} - p}.$$

So if we choose $p > \max\{\frac{m}{\tau}, 2\}$, then the sum on the right hand side is convergent and therefore we have a uniform bound for the sum on the left hand side and hence for $A(\hbar)$. This finishes the proof of (72).

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
E-mail address: `hhezari@math.jhu.edu`